

## Exercise 1

In Exercises 1 through 4, take the indented contour in Fig. 101 (Sec. 82).

Derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b - a) \quad (a \geq 0, b \geq 0).$$

Then, with the aid of the trigonometric identity  $1 - \cos(2x) = 2 \sin^2 x$ , point out how it follows that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

### Solution

Despite the fact that a singularity exists at  $x = 0$ , which lies in the interval of integration, the integral converges because the singular point is removable. That is, if the integrand is expanded about  $x = 0$ ,

$$\begin{aligned} \frac{\cos(ax) - \cos(bx)}{x^2} &= \frac{\left[1 - \frac{(ax)^2}{2} + \frac{(ax)^4}{24} - \dots\right] - \left[1 - \frac{(bx)^2}{2} + \frac{(bx)^4}{24} - \dots\right]}{x^2} \\ &= \frac{-\frac{a^2+b^2}{2}x^2 + \frac{a^4-b^4}{24}x^4 + \dots}{x^2} \\ &= -\frac{a^2+b^2}{2} + \frac{a^4-b^4}{24}x^2 + \dots, \end{aligned}$$

there are no negative powers of  $x$ . The integrand is an even function, so the interval of integration can be extended to  $(-\infty, \infty)$  as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2} dx = \int_{-\infty}^{\infty} \frac{\cos(ax) - \cos(bx)}{2x^2} dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,  $(e^{iaz} - e^{ibz})/2z^2$ , and the contour in Fig. 101.

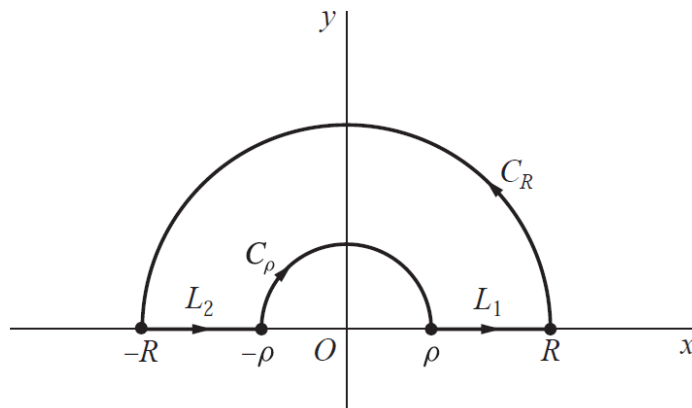


FIGURE 101

Figure 1: The singularity at  $z = 0$  is avoided by following the semicircular arc  $C_\rho$ .

According to Cauchy's residue theorem, the integral of this function around the closed contour is equal to  $2\pi i$  times the sum of the residues at the enclosed singularities  $z_n$ .

$$\oint_C \frac{e^{iaz} - e^{ibz}}{2z^2} dz = 2\pi i \sum_n \operatorname{Res}_{z=z_n} \frac{e^{iaz} - e^{ibz}}{2z^2}$$

There are no singularities other than  $z = 0$ , so the right side simplifies to zero.

$$\oint_C \frac{e^{iaz} - e^{ibz}}{2z^2} dz = 0$$

This closed loop integral is the sum of four integrals, one over each arc in the loop.

$$\int_{L_1} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{L_2} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz = 0$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L_1 : \quad z &= re^{i0}, & r = \rho &\rightarrow r = R \\ L_2 : \quad z &= re^{i\pi}, & r = R &\rightarrow r = \rho \\ C_\rho : \quad z &= \rho e^{i\theta}, & \theta = \pi &\rightarrow \theta = 0 \\ C_R : \quad z &= R e^{i\theta}, & \theta = 0 &\rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} 0 &= \int_\rho^R \frac{e^{iare^{i0}} - e^{ibre^{i0}}}{2(re^{i0})^2} (dr e^{i0}) + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_R^\rho \frac{e^{iare^{i\pi}} - e^{ibre^{i\pi}}}{2(re^{i\pi})^2} (dr e^{i\pi}) + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \\ &= \int_\rho^R \frac{e^{iar} - e^{ibr}}{2r^2} dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_R^\rho \frac{e^{-iar} - e^{-ibr}}{2(-r)^2} (-dr) + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \\ &= \int_\rho^R \frac{e^{iar} - e^{ibr}}{2r^2} dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{2r^2} dr + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \\ &= \int_\rho^R \frac{e^{iar} + e^{-iar} - e^{ibr} - e^{-ibr}}{2r^2} dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \\ &= \int_\rho^R \frac{(2 \cos ar) - (2 \cos br)}{2r^2} dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \\ &= \int_\rho^R \frac{\cos ar - \cos br}{r^2} dr + \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz. \end{aligned}$$

Take the limit now as  $\rho \rightarrow 0$  and  $R \rightarrow \infty$ . The integral over  $C_\rho$  tends to  $(-\pi/2)(b-a)$ , and the integral over  $C_R$  tends to zero. Proof for these statements will be given at the end.

$$\int_0^\infty \frac{\cos ar - \cos br}{r^2} dr + \left(-\frac{\pi}{2}\right)(b-a) = 0$$

Change the dummy integration variable to  $x$  and solve for the integral. Therefore,

$$\boxed{\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a).}$$

Note that this formula holds for  $a \geq 0$  and  $b \geq 0$ , so to obtain the second result, set  $a = 0$  and  $b = 2$ .

$$\int_0^{\infty} \frac{\cos 0 - \cos 2x}{x^2} dx = \frac{\pi}{2} (2)$$

$$\int_0^{\infty} \frac{1 - \cos 2x}{x^2} dx = \pi$$

Apply the provided trigonometric identity  $1 - \cos 2x = 2 \sin^2 x$ .

$$\int_0^{\infty} \frac{2 \sin^2 x}{x^2} dx = \pi$$

Divide both sides by 2. Therefore,

$$\boxed{\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.}$$

The Integral Over  $C_\rho$ 

Our aim here is to show that the integral over  $C_\rho$  tends to  $(-\pi/2)(b-a)$  in the limit as  $\rho \rightarrow 0$ . The parameterization of the small semicircular arc in Fig. 101 is  $z = \rho e^{i\theta}$ , where  $\theta$  goes from  $\pi$  to 0.

$$\begin{aligned}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz &= \int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2(\rho e^{i\theta})^2} (\rho i e^{i\theta} d\theta) \\ &= \int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}} (i d\theta)\end{aligned}$$

In the limit as  $\rho \rightarrow 0$ , we have

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz = \lim_{\rho \rightarrow 0} \int_\pi^0 \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}} (i d\theta).$$

Since the limits of integration are constant, the limit may be brought inside the integral.

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz = \int_\pi^0 \lim_{\rho \rightarrow 0} \frac{e^{ia\rho e^{i\theta}} - e^{ib\rho e^{i\theta}}}{2\rho e^{i\theta}} (i d\theta)$$

Plugging in  $\rho = 0$  yields the indeterminate form  $0/0$ , so l'Hôpital's rule will be applied to calculate the limit.

$$\begin{aligned}\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz &\stackrel{\frac{0}{0}}{=} \int_\pi^0 \lim_{\rho \rightarrow 0} \frac{ia e^{i\theta} e^{ia\rho e^{i\theta}} - ib e^{i\theta} e^{ib\rho e^{i\theta}}}{2e^{i\theta}} (i d\theta) \\ &= \int_\pi^0 \lim_{\rho \rightarrow 0} \frac{ia e^{ia\rho e^{i\theta}} - ib e^{ib\rho e^{i\theta}}}{2} (i d\theta) \\ &= \int_\pi^0 \frac{ia - ib}{2} (i d\theta) = \int_\pi^0 \frac{b-a}{2} d\theta\end{aligned}$$

Therefore,

$$\lim_{\rho \rightarrow 0} \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{2z^2} dz = -\frac{\pi}{2}(b-a).$$

The Integral Over  $C_R$ 

Our aim here is to show that the integral over  $C_R$  tends to zero in the limit as  $R \rightarrow \infty$ . The parameterization of the large semicircular arc in Fig. 101 is  $z = R e^{i\theta}$ , where  $\theta$  goes from 0 to  $\pi$ .

$$\begin{aligned}\int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz &= \int_0^\pi \frac{e^{iaR e^{i\theta}} - e^{ibR e^{i\theta}}}{2(R e^{i\theta})^2} (R i e^{i\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iaR e^{i\theta}} - e^{ibR e^{i\theta}}}{2R e^{i\theta}} (i d\theta) \\ &= \int_0^\pi \frac{e^{iaR(\cos\theta + i\sin\theta)} - e^{ibR(\cos\theta + i\sin\theta)}}{2R e^{i\theta}} (i d\theta) \\ &= \int_0^\pi \frac{e^{iaR\cos\theta} e^{-aR\sin\theta} - e^{ibR\cos\theta} e^{-bR\sin\theta}}{2R e^{i\theta}} (i d\theta)\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
 \left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| &= \left| \int_0^\pi \frac{e^{iaR \cos \theta} e^{-aR \sin \theta} - e^{ibR \cos \theta} e^{-bR \sin \theta}}{2Re^{i\theta}} (i d\theta) \right| \\
 &\leq \int_0^\pi \left| \frac{e^{iaR \cos \theta} e^{-aR \sin \theta} - e^{ibR \cos \theta} e^{-bR \sin \theta}}{2Re^{i\theta}} (i) \right| d\theta \\
 &= \int_0^\pi \frac{|e^{iaR \cos \theta} e^{-aR \sin \theta} - e^{ibR \cos \theta} e^{-bR \sin \theta}|}{|2Re^{i\theta}|} |i| d\theta \\
 &= \int_0^\pi \frac{|e^{iaR \cos \theta} e^{-aR \sin \theta} - e^{ibR \cos \theta} e^{-bR \sin \theta}|}{2R} d\theta \\
 &\leq \int_0^\pi \frac{|e^{iaR \cos \theta} e^{-aR \sin \theta}| + |e^{ibR \cos \theta} e^{-bR \sin \theta}|}{2R} d\theta \\
 &= \int_0^\pi \frac{e^{-aR \sin \theta} + e^{-bR \sin \theta}}{2R} d\theta
 \end{aligned}$$

So we have

$$\left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| \leq \int_0^\pi \frac{e^{-aR \sin \theta} + e^{-bR \sin \theta}}{2R} d\theta.$$

Take the limit of both sides as  $R \rightarrow \infty$ .

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{e^{-aR \sin \theta} + e^{-bR \sin \theta}}{2R} d\theta$$

Because the limits are constant, the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| \leq \int_0^\pi \lim_{R \rightarrow \infty} \frac{e^{-aR \sin \theta} + e^{-bR \sin \theta}}{2R} d\theta$$

Since  $\theta$  lies between 0 and  $\pi$ , the sine of  $\theta$  is positive. In addition,  $R$ ,  $a$ , and  $b$  are nonnegative, so each of the exponential functions tends to zero (or to one if  $a = 0$  or  $b = 0$ ). In any case, the integral tends to zero because of the  $R$  in the denominator.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{iaz} - e^{ibz}}{2z^2} dz = 0.$$